NUMERICAL SOLUTION OF A NONLINEAR REVERSE PROBLEM OF HEAT CONDUCTION

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Two numerical methods are shown for determining the transient thermal fluxes and temperatures at a surface of a heated plate with variable thermophysical properties. The effect of input data precision on the recovery of boundary functions is also considered.

We consider a reverse problem of heat conduction in a thick and infinitely large plate. At both the outer and the inner surface of the plate appear thermal fluxes variable with time $q(\tau)$ and $q_{bs}(\tau)$ respectively. Thermal flux $q(\tau)$ is unknown, but the temperature $T_0(\tau)$ at a fixed point inside the body is known.

The original τ–x space (Fig. 1) will be subdivided into two subspaces $D_1\{-x_{bs} \le x \le 0,\ 0 \le \tau \le \tau_m\}$ and $D_2\{0 \le x \le x_0,\ 0 \le \tau \le \tau_m\}$. Establishing the temperature $T_w(\tau)$ and the thermal flux $q(\tau)$ at the outer surface of the body may be regarded as extending the solution to the parabolic equation along the x-coordinate to the space boundary. The equation of heat conduction can be integrated in two different ways. In one way the integration over region D_2 proceeds in the direction of positive x, in the other way the integration over regions D_1 and D_2 proceeds in the same direction (of positive x) so that it is not necessary to subdivide the entire region. In order to evaluate each method, we have developed numerical algorithms and performed computer experiments. Let us consider the first method (scheme Num_1) first.

From the solution to the forward problem of heat conduction with stipulated boundary conditions for region D_1, one determines the temperature field and the thermal flux $q_0(\tau)$ at point x = 0. In order to start integrating in region D_2, one must know the temperature $T_0(\tau)$ and the thermal flux $q_0(\tau)$, and it is necessary to stipulate the boundary condition at $\tau = \tau_m$:

$$C\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right), \tag{1}$$

$$\left.\begin{aligned} T(x,\ 0) &= f(x), \quad \frac{\partial T(x,\ \tau_m)}{\partial \tau} = \varphi(x),\\ T(0,\ \tau) &= T_0(\tau), \quad -\lambda\frac{\partial T(0,\ \tau)}{\partial x} = q_0(\tau). \end{aligned}\right\} \tag{2}$$

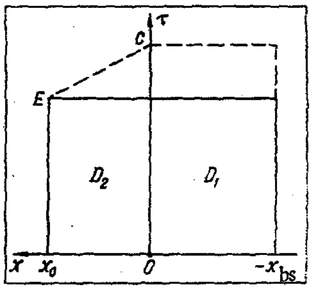

Fig. 1. Regions defined for the solution of the forward and the reverse problem of heat conduction.

The end result of solving this problem is to be the determination of the unknown functions $T_w(\tau)$ and $q(\tau)$. We note that it is not usually possible to stipulate function $\varphi(x)$ a priori. As has been shown in computer experiments, one may let $(\partial T(x,\ \tau_m))/\partial\tau = 0$ or, better still, $(\partial T(x,\ \tau_m))/\partial\tau = (\partial T(0,\ \tau_m))/\partial\tau = \text{const}$. The solution deviates then from the sought solution within a rather narrow region $\bar{\tau} < \tau \le \tau_m$. This deviation can be reduced significantly, if an approximation to function $T(x,\ \tau_m)$ is found by extrapolating in each x_i layer from known temperatures T_{in} (n < m) in accordance with the selected grid $\{x_i\}i = 0,\ 1,\ \ldots,\ k;\ \{\tau_n\}n = 1,\ 2,\ \ldots,\ m$.

As a rule, however, the necessity of stipulating the boundary condition at $\tau = \tau_m$ can be avoided but, consequently, the solution in the vicinity of this boundary will become less precise. This is possible because of the temperature $T_0(\tau)$ being usually known not only through the interval $[0,\ \tau_m]$

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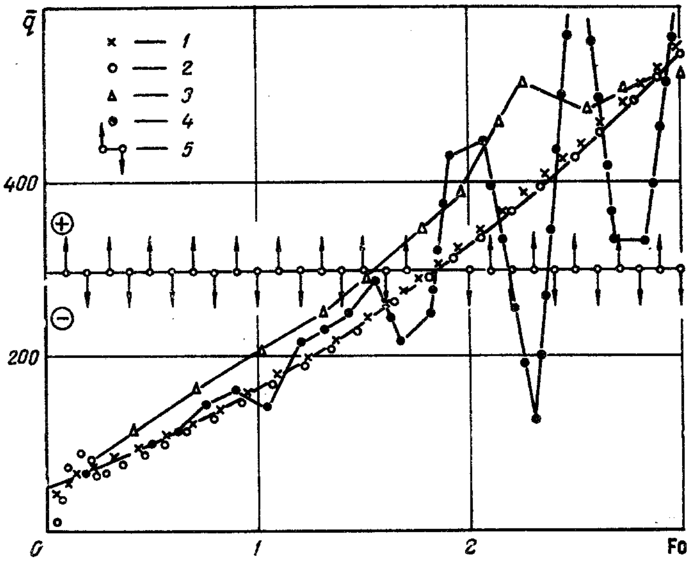

Fig. 2. Results of solving a methodological reverse problem with exact and with perturbed values of the input temperature ($\bar{q}$, °C; $x_0 = 0.8$ mm): solid lines represent exact solutions; 1) algorithm of Num_1 with $\delta \sim 0$, $\Delta Fo = 0.06$, and $k = 8$; 2) algorithm of Num_2 with $\delta \sim 0$, $\Delta Fo = 0.06$, and $\varkappa = 5$; 3) algorithm of Num_1 with a normal distribution of temperature errors $3\sigma_n(\tau) = 0.1T_0(\tau)$, $\Delta Fo = 0.32$, and $k = 6$; 4) algorithm of Num_1 with $3\sigma_n(\tau) = 0.1T_0(\tau)$, $\Delta Fo = 0.15$, and $k = 6$); 5) algorithms of Num_1 and Num_2 with $3\sigma_n(\tau) = 0.05T_{0,max}$ and $\Delta F0 = 0.06$.

but also at $\tau > \tau_m$, which permits a modification of region D_2 from a rectangular to a trapezoidal one (Fig. 1). The temperatures at the free boundary EC are obtained automatically in the course of the solution process.

We now introduce a new variable $\theta = (1/\lambda_0)\int_0^T \lambda(T)dT$. Problems (1)-(2) becomes then

$$\frac{\partial\theta}{\partial\tau} = a(\theta)\frac{\partial^2\theta}{\partial x^2}, \tag{3}$$

$$\theta(x,\ 0) = \bar{f}(x), \quad \frac{\partial\theta(x,\ \tau_m)}{\partial\tau} = \bar{\varphi}(x),$$

$$\theta(0,\ \tau) = \theta_0(\tau), \quad -\lambda_0\frac{\partial\theta(0,\ \tau)}{\partial x} = q_0(\tau).$$

The difference analog of Eq. (3) is constructed according to the explicit symmetric "criss-cross" scheme:

$$\theta_{i+1n} = 2\theta_{in} - \theta_{i-1n} + \frac{(\Delta x)^2}{2a_{in}\Delta\tau}(\theta_{in+1} - \theta_{in-1}).$$

The thermal flux is determined from the relation based on the method of elementary heat balances:

$$q_n = \lambda_0\frac{\theta_{kn} - \theta_{k-1n}}{\Delta x} + \frac{\lambda_0}{a_{in}}\frac{\Delta x}{2}\frac{\theta_{kn+1} - \theta_{kn}}{\Delta\tau}.$$

We will next consider the second method of solving this reverse problem of heat conduction (scheme Num_2). The solution is constructed here according to the implicit finite-difference scheme for approximating the quasilinear equation of heat conduction, written as follows (the origin of coordinates is located on the plate boundary):

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial\tau} - \frac{\lambda'}{\lambda}\left(\frac{\partial T}{\partial x}\right)^2, \quad 0 < x < b, \quad \tau > 0, \tag{4}$$

where

$$\lambda = \lambda(T); \quad a = a(T); \quad \lambda' \equiv \frac{d\lambda}{dT}.$$

We thus write for (4) an implicit six-point scheme, the second derivative weighted with $\sigma = 1$ and approximated (see, for instance, [1]):

$$\begin{gathered} -\frac{a_{in}\Delta\tau}{(\Delta x)^2} T_{i-1n+1} + \left[1 + \frac{2a_{in}\Delta\tau}{(\Delta x)^2}\right] T_{in+1} - \frac{a_{in}\Delta\tau}{(\Delta x)^2} T_{i+1n+1} \\ = T_{in} + \frac{a_{in}\Delta\tau}{(\Delta x)^2}\left(\frac{\lambda'_{in}}{4\lambda_{in}}\right)(T_{i+1n} - T_{i-1n})^2. \end{gathered} \tag{5}$$

Assuming, for simplicity, that the back surface of the plate is thermally insulated ($q_{bs} = 0$), we write the boundary conditions at $x = 0$ and $x = b$:

$$q_{n+1} - \lambda_{0n}\frac{T_{0n+1} - T_{1n+1}}{\Delta x} = C_{0n}\frac{\Delta x}{2}\cdot\frac{T_{0n+1} - T_{0n}}{\Delta\tau}, \tag{6}$$

$$-\lambda_{ln}\frac{T_{ln+1} - T_{l-1n+1}}{\Delta x} = C_{ln}\frac{\Delta x}{2}\frac{T_{ln+1} - T_{ln}}{\Delta\tau}, \tag{7}$$

respectively. The initial condition is

$$T_{i0} = f(x_i), \quad i = 0, 1, \ldots, l.$$

Relations (5)-(7) yield a system of linear algebraic equations for the unknown thermal flux q_{n+1} and the unknown temperatures T_{in+1} ($i = 0, \ldots, \varkappa-1, \varkappa+1, \ldots, l$, where $T_\varkappa$ is a known temperature at a given point $i = \varkappa$ inside the body). As a result, with the notation

$$d_{in} = 1 + 2\frac{a_{in}\Delta\tau}{(\Delta x)^2}, \quad e_{in} = -\frac{a_{in}\Delta\tau}{(\Delta x)^2},$$

$$b_{in} = T_{in} + \frac{a_{in}\Delta\tau}{(\Delta x)^2}\left(\frac{\lambda'_{in}}{4\lambda_{in}}\right)(T_{i+1n} - T_{i-1n})^2,$$

$$f_{0n} = \frac{2a_{0n}\Delta\tau}{\lambda_{0n}\Delta x} q_{n+1} + T_{0n},$$

$$Az = b,$$

we have

$$A = \begin{bmatrix} d_{0n} & 2e_{0n} & & & & & & & & -1 \\ e_{1n} & d_{1n} & e_{1n} & & & & & \mathbf{0} & & 0 \\ & e_{2n} & d_{2n} & e_{2n} & & & & & & 0 \\ & & & \ddots & & & & & & \vdots \\ & & & e_{\varkappa-2n} & d_{\varkappa-2n} & e_{\varkappa-2n} & & & & 0 \\ & & & & e_{\varkappa-1n} & d_{\varkappa-1n} & & & & 0 \\ & & & & & e_{\varkappa n} & e_{\varkappa n} & & & 0 \\ & & & & & & d_{\varkappa+1n} & e_{\varkappa+1n} & & 0 \\ & \mathbf{0} & & & & & e_{\varkappa+2n} & d_{\varkappa+2n} & e_{\varkappa+2n} & 0 \\ & & & & & & & & \ddots & \vdots \\ & & & & & & & 2e_{ln} & d_{ln} & 0 \end{bmatrix},$$

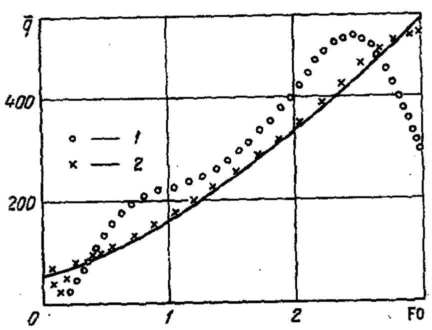

Fig. 3. Results of solving a methodological reverse problem according to scheme Num_2, with smoothed values of the input temperature ($\bar{q}$, °C; $x_0 = 0.08$ mm): solid line represents exact solution; 1) normal distribution of temperature errors with $3\sigma_n(\tau) = 0.1T_{\varkappa,max}$, $\Delta Fp = 0.06$, and $\varkappa = 5$ ($T_\varkappa$ smoothed by fourth differences, number of smoothings N = 10,000); 2) $3\sigma_n(\tau) = 0.1T_{\varkappa,max}$, $\Delta Fo = 0.06$, and $\varkappa = 5$ ($T_\varkappa$ recovered by the regularization method).

$$
z = \begin{bmatrix} T_{0n+1} \\ T_{1n+1} \\ T_{2n+1} \\ \cdot \\ \cdot \\ \cdot \\ T_{\varkappa-2n+1} \\ T_{\varkappa-1n+1} \\ T_{\varkappa+1n+1} \\ T_{\varkappa+2n+1} \\ T_{\varkappa+3n+1} \\ \cdot \\ \cdot \\ f_{0n} \end{bmatrix}; \qquad b = \begin{bmatrix} 0 \\ b_{1n} \\ b_{2n} \\ \cdot \\ \cdot \\ \cdot \\ b_{\varkappa-2n} \\ b_{\varkappa-1n} - e_{\varkappa-1n} T_{\varkappa n+1} \\ b_{\varkappa n} - d_{\varkappa n} T_{\varkappa n+1} \\ b_{\varkappa+1n} - e_{\varkappa+1n} T_{\varkappa n+1} \\ b_{\varkappa+2n} \\ \cdot \\ \cdot \\ T_{ln} \end{bmatrix}.
$$

The solution to the system Az = b can be based on the Gauss elimination method or on the square-root method [2]. In the latter case the original system is reduced, with the aid of a Gauss transformation, to a system with a positive-definite matrix

$$A'Az = A'b.$$

Numerical calculations by both methods have yielded almost identical answers to problems of this kind.

On the basis of these algorithms for solving reverse problems, programs were written in the ALGOL language and methodological examples were solved on two computers: model M-220 and model BESM-6. The results are shown here for one such example. For the purpose of analysis, the body was considered to be an excellent thermal insulator ($\lambda = 1.3 \cdot 10^{-4}$ kW/m · °C and $a = 1.2 \cdot 10^{-7}$ m²/sec). The thermal flux function was known a priori ($\bar{q} = qx_0/\lambda$, $Fo = a\tau/x_0^2$):

$$\bar{q} = 48 + 88\,Fo + 27\,Fo^2, \text{ °C}.$$

From the solution to the forward problem of heat conduction in region D_1, we found the values of the temperature at point x_0 which corresponded to $\bar{q}$. An accuracy within $\delta \leq 10^{-2}$ to 10^{-3}°K was attained. The temperature thus found was then adopted as the exact input function for solving the reverse problem of heat conduction. The sought $T_W(\tau)$ and $q(\tau)$ curves determined according to schemes Num_1 and Num_2 were stable within the given range $0.06 \leq \Delta Fo \leq 0.32$ (Fig. 2). However, random perturbations in the initial temperature produced instabilities in the computation process at Fo < 0.3-0.4 (Fig. 2). This is applied particularly to the algorithm of scheme Num_2. A necessary increase of the $\Delta\tau$ step may significantly worsen the accuracy of the solution. In many problems, moreover, where intensive thermal phenomena are simulated experimentally the $\Delta\tau$ step becomes prohibitively large (sometimes even larger than the entire test time interval of interest to the researcher) and such a method of stabilizing the solution is unsuitable. For this reason, we studied the feasibility of specially processing the input data so as to produce stable approximations at sufficiently small $\Delta\tau$ steps.

The input data were smoothed by the fourth-differences (five points) method and the input function was recovered by the regularization method.

Smoothing by means of fourth differences [3] is based on the formula for successive (point-to-point) refinements of the central values (at j = 0) of second-degree polynomials $T = a + bj + cj^2$ drawn through five points according to the principle of least squares:

$$a = T_0 - \frac{3}{35}\delta^4 T_0.$$

This method usually yields a sufficiently smooth curve within a short machine time.

Smoothing the input data yields only some average approximation to the sought temperature curve $T(\tau)$ (without recovering the derivative $T'(\tau)$). Numerical experiments have shown that this method of generating the input function is advantageous when the fluctuation errors in the values of the temperature remain small. In real experiments these errors are due to the inaccuracy of measuring, recording, and decoding devices. When the fluctuation errors become sufficiently large, then the application of this method to the solution of reverse problems may, in the final analysis, lead to an appreciable distortion of the boundary conditions (Fig. 3). When generating the input data in those cases, one must seek the approximation not only to the sought function but also to its derivative.

Determining the derivative of a function on the basis of test data is an ill-stated problem, because of the unboundedness of the differential operator. An algorithm for uniformly approaching a derivative can be constructed on the basis of A. N. Tikhonov's general method of solving ill-stated problems [4]. In the course of solving reverse problems of heat conduction, the temperature data are preliminarily processed by the following simple algorithm for the recovery of the derivative $T'(\tau)$

We consider the problem of solving an integral equation with an approximate right-hand side:

$$Au \equiv \int_0^\tau u(\xi)\,d\xi = \vartheta(\tau),$$

where

$$u(\tau) = \frac{dT}{d\tau}, \quad \vartheta(\tau) = T(\tau) - T(0).$$

we approximate the integral by

$$\vartheta(\tau_n) = \sum_{i=1}^{n} \bar{u}_i \Delta\tau,$$

where

$$\bar{u}_i = \frac{u_{i-1} + u_i}{2},$$

and we select the following regularizing functional [4]:

$$\Phi_\alpha[u] = \sum_{n=1}^{m} \left\{ \sum_{i=1}^{n} \bar{u}_i \Delta\tau - \vartheta_n \right\}^2 \Delta\tau + \alpha \sum_{i=0}^{m} \frac{(\bar{u}_{i+1} - \bar{u}_i)^2}{\Delta\tau}, \tag{8}$$

with the regularization parameter $\alpha > 0$.

Equating the partial derivatives of $\bar{u}_i$ to zero and assuming, for simplicity, zero boundary values: $u'(0) = u'(\tau_m) = 0$ (which, generally, is not absolutely necessary), we obtain a linear system of algebraic equations for the family of discrete functions $\bar{u}_\alpha(\tau)$ minimizable by means of the functional Φ_α (8):

$$\sum_{l=1}^{m} a_{lk} \bar{u}_l = f_k, \quad k = 1, 2, \ldots, m; \tag{9}$$

where

$$a_{lk} = \Delta\tau(m - l + 1), \quad l \geqslant k + 2;$$
$$a_{lk} = \Delta\tau(m - l + 1) - \alpha, \quad l = k + 1;$$
$$a_{ll} = \Delta\tau(m - l + 1) + 2\alpha, \quad l \neq 1;\ m;$$
$$a_{ll} = \Delta\tau(m - l + 1) + \alpha, \quad l = 1;\ m;$$
$$f_k = \sum_{n=k}^{m} \vartheta_n.$$

System (9) was solved by the square-root method, permitting very precise calculations in a relatively short machine time. The best approximation to the derivative was selected on the basis of the divergence principle, in the form of the equality in [5]. We minimized the quantity

$$\Delta_1 = \left| \left[\sum_{n=1}^{m} \left(\sum_{i=1}^{n} \bar{u}_i \Delta\tau - \vartheta_n \right)^2 \Delta\tau \right]^{\frac{1}{2}} - \left[\int_0^{\tau_m} \sigma^2(\tau)\,d\tau \right]^{\frac{1}{2}} \right|,$$

with $\sigma(\tau)$ denoting the absolute mean-squared error of measurements. The function $\sigma(\tau)$ had to be stipulated. In practice it is usually necessary to compute discrete values of statistical estimates of this quantity which correspond to fixed instants of time at some $\Delta\tau$ step. For this, of course, one needs a sufficiently broad selection of random realizations of the input function.

An input function $T(\tau)$ constructed on the basis of the recovered derivative closely approximated the exact function and yielded a close approximation to the sought boundary condition (Fig. 3).

In many cases the derivative of a function has no independent value and only the input function is to be recovered from $T(\tau)$ data. This problem can be solved even more simply. It may, for example, correspond to the problem of minimizing the functional

$$\Phi_\alpha [T] = \sum_{n=1}^{m} (T_n - T_{\delta n})^2 \, \Delta\tau + \alpha \sum_{n=0}^{m} \frac{(T_{n+1} - T_n)^2}{\Delta\tau} .$$

In this way, an "improvement" of the input data permits a significant reduction of the critical step size $\Delta\tau_{cr}$ at which instability of results becomes pronounced.

The algorithms for solving nonlinear reverse problems can be easily extended to the case of a composite (e.g., two-layer) plate. Without any theoretical difficulties, they can also be converted for solving problems where the heat transmission through a body is described by the general equation of heat conduction

$$C \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x} \right) + m \frac{\partial T}{\partial x} + Q,$$

with internal gas and heat sources present in the body. The algorithm of Num_1 will not change significantly, if the boundary with an unknown condition becomes movable (e.g., as a result of material wear). A change will occur only in the shape of the integration region in which the movable boundary is defined by a stipulated law $X(\tau)$.

NOTATION

A	is a matrix or an operator;
A'	is the transposed matrix;
a	is the thermal diffusivity;
b	is the plate thickness;
C	is the specific heat referred to volume;
D_1, D_2	are the integration regions for the equation of heat conduction;
f(x)	is the initial temperature distribution;
q	is the unknown thermal flux;
q_{bs}	is the thermal flux at the inner (back) surface;
T	is the temperature;
T_w	is the temperature at the outer surface;
$T_{\delta n}$	are perturbed values of the input temperatures;
x	is the space coordinate;
Δx	is the step of numerical integration along x;
Fo	is the Fourier number;
ΔFo	is the increment in the Fourier number;
α	is the regularization parameter;
δ	is the error of input temperature values;
θ	is the model temperature defined by a Kirchhoff transform;
λ	is the thermal conductivity;
τ	is time;
τ_m	is the right-hand boundary of the time interval;
$\Delta\tau$	is the step of numerical integration along the time τ;
τ_n, x_i	are points on the time—space grid;
$\varphi(x)$	is an additional boundary condition for solving a reverse problem by the scheme Num_1.

Subscript

0 refers to values at point x = 0.

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